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TECHNICAL REPORT

Dynamic construals, static formalisms: Evidence from co-speech gesture during mathematical proving

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DYNAMIC CONSTRUALS, STATIC FORMALISMS: EVIDENCE FROM CO-SPEECH GESTURE DURING MATHEMATICAL PROVING[1]

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Abstract

This paper presents results from ongoing research using co-speech gesture to investigate the nature of mathematical concepts. Traditional accounts of the content of mathematical concepts have focused on formal language and definitions. In analysis, key concepts such as continuity are defined by means of static existential and universal quantifiers ranging over static numbers. Recent evidence from Cognitive Science, however, suggests that mathematical concepts are construed through various mechanisms of everyday cognition — such as conceptual metaphor and fictive motion — which are at odds with static conceptions. We analyse the co-speech gesture produced by graduate students while collaborating on a proof in analysis. The results support the claim that many mathematical concepts, which formally make use of static entities and relations, are, cognitively, inherently dynamic.

INTRODUCTION

Advanced mathematics is characterized by rigorous methods and symbolic manipulation. Traditionally, this has been taken as evidence for the abstract, formal nature of mathematical concepts. Contrary to this view, recent research from Cognitive Science suggests that nature of such concepts is largely embodied, and so mathematical concepts — such as infinity, continuity, number, and the rest — are not abstract and generated from formal definitions, but metaphorical and grounded in experience [17]. The evidence for the embodiment of mathematics has been drawn largely from mathematical language, using techniques in Cognitive Linguistics, although recent research has included some qualitative studies of gesture [21]. This paper reports on the results of an ongoing research project on the nature of mathematical proof. In particular, we focus on the co-speech gesture produced by graduate mathematics students while collaborating on a proof, and argue that the character of these gestures supports the claim that mathematical concepts are largely metaphorical and embodied and that their nature cannot be reduced to pure formalisms. This research extends earlier work that used gesture to adjudicate the cognitive reality of conceptual metaphor in mathematics [21, 22], moving beyond the pedagogical setting to consider advanced mathematical practice in a naturalistic setting.

The paper is organized as follows. First, we briefly review the evidence for metaphor and fictive motion in mathematical cognition. Second, we introduce the study of gesture as a tool for investigating the cognitive reality of these phenomena during mathematical practice. Then we present some quantitative results on co-speech gesture production from a semi-structured case study of collaborative proof by mathematics graduate students at a large research university. Finally, we discuss implications for the nature of mathematical concepts.

DYNAMISM IN MATHEMATICAL DISCOURSE

How can we, as limited beings, understand abstract concepts that transcend our human experience? According to some approaches in Embodied Cognition, the answer lies largely in our shared embodiment, in the way our bodies — modulated by culture — structure thought and experience. Research in Cognitive Science suggests that abstract concepts are created and understood via a number of fundamental cognitive processes that extend the inferential structure of our bodily experiences. These processes include conceptual metaphor and metonymy [16], conceptual blending [4], and fictive motion [30]. Using the tools of Cognitive Linguistics, Lakoff and Núñez [17] argued that even mathematical concepts rely on these cognitive processes. While the details of the various construals underlying mathematical thought are beyond the scope of this article, in this section we will quickly review a few salient instances in which fictive motion lends dynamism to putatively static mathematical concepts.

[1] A previous version of this work was included in the *Proceedings of the International Symposium on Mathematical Practice and Cognition*, Alison Pease, Markus Guhe, and Alan Smaill (Eds.), at the AISB 2010 convention, 29 March – 1 April 2010, De Montfort University, Leicester, UK.

Consider the concept of a limit, a central notion in calculus. Formally, the limit of a function is defined by a chain of inequalities:

> Let a function f be defined on an open interval containing a, except possibly at a itself, and let L be a real number. Then $\lim_{x \mapsto a} f(x) = L$ means that, for all $\epsilon > 0$, there exists $\delta > 0$, such that whenever $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Note that the limit notation includes a small arrow, which might suggest that the definition of a limit of a function would include some form of dynamism. The formal definition of a limit, however, refers only to static universal and existential quantifiers, static numbers, motion-less arithmetic difference and static inequalities. Nowhere in this definition is there any mention of movement. Mathematicians, on the other hand, speak of a function "tending to," "moving toward" or "reaching" a limit — all of which, contra the formal definition, invoke a sense of motion [21]. These expressions are a form of fictive motion [30], the process by which we unconsciously conceptualize static entities in dynamic terms.

Fictive motion construals always involve the motion of a *trajector* across a *landscape*. When we say, for instance, that "the Equator passes through Brazil," the Equator — a purely imaginary static entity — is construed as a moving agent (trajector) dynamically crossing a country (landscape). Similarly, we can say that a fence *stops* at a tree or that a road *runs* along the coast — even though both fences and roads are completely stationary and thus incapable of stopping or running.

This same cognitive mechanism of fictive motion injects dynamism into a wide range of statically defined mathematical entities. A function, for example, is formally defined as a static relation between two sets, the domain and the range, but mathematicians nevertheless describe functions dynamically as "reaching a limit," "going down towards a minimum," or "oscillating," in each case evoking a construal in which an imaginary trajector travels along the path of the function [21]. Fictive motion is similarly at work when we say that sequences are "approaching," "decreasing," or "converging," and when arithmetic is construed as motion along a number-line.[1] Dynamism, therefore, is present throughout the language of mathematics — showing up in the discourse surrounding continuity, functions, and even arithmetic — and lends credence to the claim that mathematical thought itself is dynamic, metaphorical, and embodied.

MATHEMATICAL GESTURE

One objection to this line of reasoning is that these metaphorical construals are mandatory in mathematical discourse — and that, therefore, they are conventionalized, dead, stripped of any cognitive reality [21]. Consider the geometrical procedure of "reflection," such as when a point in the Cartesian plane is *reflected across the origin*. The word "reflection" has roots in the Latin verb *reflectere*, meaning "to bend back." There is no other way of describing the geometrical procedure, and therefore the same lexical item, "reflection," is always used to describe the procedure — yet there is also no evidence suggesting that the concept of reflection involves an automatic construal of *bending backwards*. Certain aspects of mathematical discourse, then, are codified and devoid of cognitive significance. Could the dynamic discourse surrounding limits, functions, and sequences similarly involve conventionalized discourse? To address this objection, we must supplement corpus studies of mathematical discourse with additional lines of converging evidence.

The cognitive reality of mathematical construals is supported by the study of gesture, that is, motor action co-produced with speech and thought in real time. Gesture is universal, unconscious, and essential to communication. Most importantly, gesture offers a "window into the mind" [8]. When co-produced with abstract thinking, gestures parallel the metaphorical mappings exhibited linguistically [19, 2, 26], and give us insight into the representation of mathematical concepts and solution strategies [1, 5]. In particular, Núñez [21, 22] demonstrated that mathematicians' gestures in pedagogical contexts supply converging evidence for the metaphorical and embodied nature of mathematical concepts.

Previous research on mathematical gesture, however, has dealt primarily with gesture production during pedagogy or in the context of elementary mathematical problem solving. In these settings, gestures were found to be dynamic — in line with the predictions of Cognitive Linguistics, and suggesting that mathematical concepts are metaphorical *in those settings* [22, 3]. But would we expect any other behavior? It is standard pedagogical practice to use "real world" examples of abstract concepts, to ground the abstruse in the everyday. Physics teachers might describe electricity, for instance, as "water running through a pipe," effectively mapping intuitions about water volume and pressure onto the more abstract concepts of electrical current and voltage [6]. The use of this pedagogical scaffold, however, does not imply that electrical current is in reality the flow of water particles. Certainly, the expert physicist may call on such metaphors while instructing a naïve student, or may subtly deploy these evocative images during heuristic reasoning, but all the while they might recognize that electrical current is *fundamentally different* from water flow. Expert practice requires the careful amendment of pedagogical metaphors. The learning of physics — one story goes — is marked by the gradual abandonment of these metaphorical construals, replacing such didactic scaffolds with genuine intuitions about basic physical phenomena [29].

Thus, while the evidence for dynamic gesture in mathematical pedagogy and communication is suggestive, it does not directly address the nature of mathematical practice — or of mathematics itself. Indeed, to date there is little research on co-speech gesture during the activities that are central to research mathematics, such as proving and communicating non-trivial results. When mathematicians are generating a proof or communicating with other expert mathematicians, do they deploy the same conceptual metaphors that are evidenced in corpus studies of mathematical discourse and in gesture studies of mathematical pedagogy? Or is the metaphorical content of these utterances an artifact of the pedagogical context?

The current study uses the tools of Gesture Studies, Cognitive Linguistics, and Embodied Cognition to empirically investigate with quantitative methods the cognitive reality of fictive motion in mathematical practice. In a semi-controlled situation, we looked at the co-speech gesture of graduate mathematics students as they collaborated in pairs on a mathematical proof involving key concepts in analysis. If the meaning of such essential mathematical concepts is determined by their formal definition, then we should expect static

[1] The number-line is itself a *conceptual blend*, the result of combining the mental spaces for number and space.

co-speech gesture. If mathematical concepts are truly metaphorical and dynamic, on the other hand, we should expect the co-speech gesture of the graduate students to reflect this dynamism.

METHOD

Participants and Procedure

Twelve graduate mathematics students from a large American research university were paid to participate. Students worked in pairs to prove a fixed point theorem:

Theorem 1 *Let f be a strictly increasing function from $[0, 1]$ to $[0, 1]$. Then there exists a number a in the interval $[0, 1]$ such that $f(a) = a$.*

We selected this problem for our study because the mathematician who proposed it reported that when proving the theorem, he experienced a palpable sense of motion.[2] Further, the problem involves crucial concepts such as limits, continuity, and increasing functions. The proving of this theorem, therefore, presents an excellent opportunity to investigate the cognitive reality of metaphor and fictive motion.

Participants had up to 40 minutes to solve the problem, working alone in a room with a blackboard. Once they were satisfied with their proof or the 40 minutes had passed, the participants explained their proof to the experimenter. The entire session was video-recorded.[3]

Coding

Based on previous research in Cognitive Linguistics [30, 25, 21], we generated a list of lexical items thought to elicit fictive motion or metaphorical construals. These included mathematical terms (e.g. function, continuity, limit, contain), verbs of motion (e.g. to cross, to move, to jump), and spatial terms (e.g. up, between, left). The gesture production of each participant was examined for representational gestures that were co-produced with lexical items on this list.

The production of gesture, as co-speech motor *action*, necessarily involves dynamic movement. To differentiate between gestures for which dynamism is an artifact of gesture production, and those gestures for which dynamism is truly expressive, we devised a coding scheme that attended to details of the motion and timing of the co-speech gesture. A gesture was coded as dynamic if it used smooth, unbroken motions; as static if it consisted of beats and segmented motions, or of a smooth motion bookended by beat gestures; and as ambiguous if it was difficult to fit into either of these categories. This coding scheme is very similar to one used in the literature, with good reliability, to classify “continuous” and “discrete” representations [1].

RESULTS AND ANALYSIS

General results and statistics

Every dyad arrived at a reasonable solution that included most of the ingredients of a complete proof. In all cases, participants’ proofs followed the same outline as the proof supplied by the mathematician who proposed the problem. Half of the dyads finished before the end of the allotted time. All six dyads make extensive use of diagrams — primarily graphs of functions — in combination with extensive symbolic inscriptions, usually involving $\epsilon - \delta$ and set notation.

Participants produced a large number of representational gestures. Every participant but one produced representational gestures that were co-timed with the specified lexical items. A total of 166 of these co-timed representational gestures were coded, for a mean of 13.8 coded gestures per participant. Of these, the majority were coded as dynamic (50.6%); slightly less were coded as static (41.6%).[4] Figure 1 summarizes gesture production by participant.

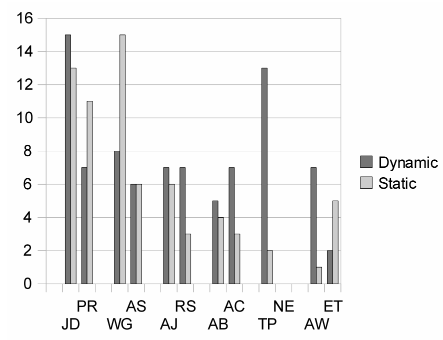

Figure 1. Breakdown of gesture production by participant. Paired participants are listed in adjoining columns.

It bears noting that the majority of gestures produced by the participants during the proof session were deictic, anchored to inscriptions on the blackboard. Deictic gestures appeared to play a number of roles, including maintaining attention during particularly complex deductions and directing the attention of a collaborator to a salient inscription. These indexical gestures, however, were ignored by the present study.

Gesture production varied according to the co-occurring concept. Certain concepts were associated with a prevalence of dynamic or static gestures. Gestures co-produced with talk of “increase,” “continuity,” and “intersection” were more often dynamic; those co-produced with talk of “containment” and “closeness” were more often static. See Figure 2.

To test the statistical significance of this result, we examined those participants who produced gestures that co-occurred with these concepts and calculated the proportion of dynamic gestures. We focused on those mathematical concepts which, based on previous theoretical analyses, were thought to have a privileged construal that was primarily dynamic or primarily static [17]. The notion of continuity, for instance, can be construed dynamically as movement without jumps – but also as “Preservation of Closeness,” a static construal [25]. The statistical analysis, therefore, focused on the dynamic notion of increase – thought to evoke fictive motion – and the static notions of closeness and containment. Among those participants who gestured while speaking of “increase,” co-produced gestures were significantly more often *dynamic* (F(1,10) = 28.90, $p = .0003$). Participants produced a significantly higher proportion of *static* gestures, on the other hand, while discussing “containment” ($F(1, 12) = 6.75, p = .0232$) and “closeness” ($F(1, 10) = 76.73, p < .0001$).

Examples of dynamic and static co-speech gesture

The following are examples of dynamic and static representational gestures co-produced with coded lexical items.

[2] We thank Guershon Harel for proposing this theorem

[3] The experimenter also conducted pre- and post-proof interviews with the participants, but these were not used in the present study.

[4] The remaining gestures were ambiguous.

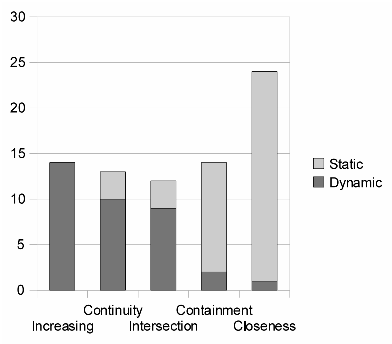

Figure 2. Number of dynamic and static gestures by co-occurring concept

Dynamic gesture

The concept of "increase" seemed to demand an exclusively dynamic treatment. Participants produced fourteen representational gestures that co-occurred with the lexical items "increasing" or "increase," all of which were dynamic.

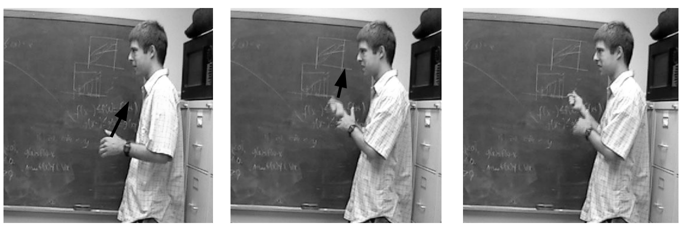

Figure 3. Example 1. Dynamic gesture produced while describing an increasing sequence. Both hands evoke the sequence's fictive motion.

In Example 1 (see Figure 3), the participant is discussing "increasing sequences" and produces a smooth, unbroken motion, co-timed with his speech. At the onset of the word "increasing," he begins to fluidly move his left hand upwards and toward the right, with his thumb pointing in the direction of motion (Figure 3, first frame). As he reaches the end of the word, the motion of his left hand slows slightly while his right hand begins to accelerate, once again moving upwards and to the right in the direction of his extended thumb. As he begins to say "sequence," his right hand reaches its top speed (Figure 3, second frame). Both hands begin to slow to a stop (Figure 3, third frame), and their retraction is co-timed with the end of "sequences." Neither his gaze nor his thumbs are directed toward a blackboard inscription.

In Example 2 (Figure 4), the student is at the blackboard, writing a series of inequalities that contradict the assumption that the function is increasing. As he finishes writing, he steps back from the blackboard, drops his hands to his side, and states, "So that contradicts uhhh increasing" (Figure 4, left frame). Precisely co-timed with the onset of "increasing," his right hand flies upwards and to the right, with his index finger extended in a prototypical pointing handshape (Figure 4, middle frame). As he finishes saying "increasing," his right hand slows, pointing to the right of the blackboard for a moment (Figure 4, right frame), before smoothly dropping to his side. This is a quintessential example of dynamic co-speech gesture, consisting of a continuous motion co-timed with a dynamic lexical item.

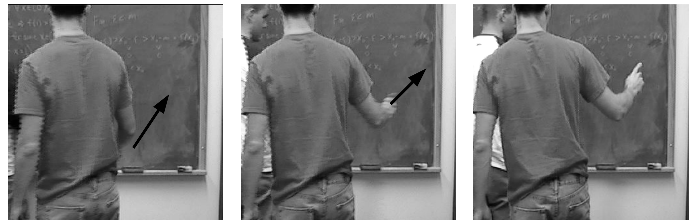

Figure 4. Example 2. Dynamic gesture during discussion of an increasing function. A fluid rightward hand movement is co-timed with speech.

Static gesture

Static gestures often accompanied discussions of closeness and containment. In Example 3 (Figure 5), the student is discussing the values of a function, and wants to consider only those values contained within a restricted region. At first, all her gestures are deictic, anchoring her discussion to the graph she has drawn on the blackboard. She introduces the region of interest by saying, "Well, if you look at a, sort of, small enough [region]." When she begins to describe the region of interest ("Well…"), she retracts her hands from the blackboard and positions them in front of her chest, pointing towards each other, and pauses as she says "sort of" (Figure 5A). Co-timed with the onset of "small," she quickly moves her hands toward each other, stopping when they are 10 cm apart, then quickly retracting her hands to their original distance (Figure 5A-C). Co-timed with the word "enough," she repeats the same inward staccato stroke, stopping abruptly when her hands are at the same distance, and retracting once again (Figure 5D-E). By indexing two particular points in space with a repeated beat, she evokes the endpoints of a delimited region containing the function's domain. This is a static representational gesture since it consists of distinct beats, indexing exact points in space.

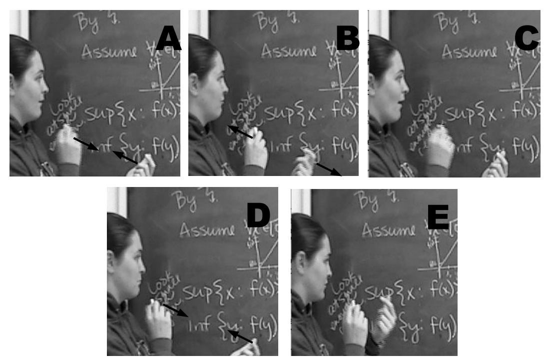

Figure 5. Example 3. A static gesture co-timed with the utterance, "small enough."

Variable gesture

Often, a single utterance was amenable to very different representations in gesture. The utterance "to the left," for instance, received both dynamic and static treatments in gesture. See Figure 6.

In the first frame of Figure 6, the participant is producing a static gesture co-timed with the utterance "to the left." One participant suggests to his collaborator that they should look for "implicit continuity to the left of what [they]'re talking about." As he says "to the left," he forms his thumb and forefinger into a U-shape, representing the interval to the left, performs one forward beat with this handshape, and then holds his hand still for nearly a full second. This gestural representation of “to the left” captures the static notion of containment.

The second frame of Figure 6 shows the same lexical affiliate (“to the left”) co-timed with a *dynamic* gesture. While discussing the intersection of a function and the line $y = x$, a participant asks if the function was “going a little bit to the left.” As he begins to say “to the left,” he points his forefinger to the left and fluidly moves his hand in that direction, retracting his hand as he says “left.” This gesture seems to represent a dynamic sense of the fictive motion for the function.

These two cases illustrate that gestures may share a lexical affiliate but differ in their dynamism. Although highly synchronous with the utterance “to the left,” these two gestures exhibit different kinetics and handshapes.

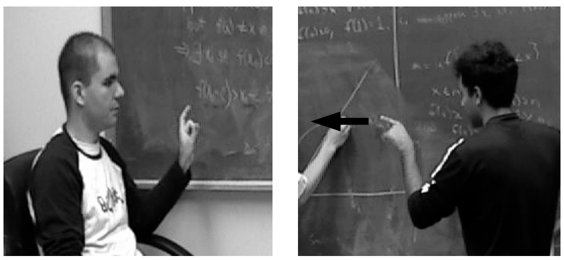

Figure 6. Example 4. Two gestures co-produced with the utterance, “to the left.” While the gesture on the left is static, the one on the right is dynamic.

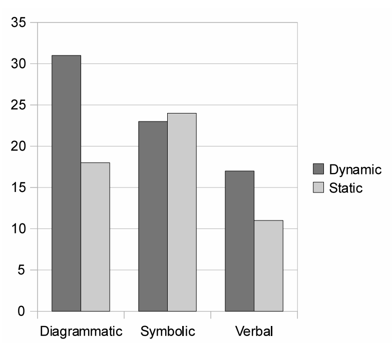

Figure 7. Number of dynamic and static gestures by mode of reasoning

Finally, gesture also varied by mode of reasoning, as documented in Figure 7. While engaged in diagrammatic reasoning — either considering or producing a diagram on the blackboard — most participants were more likely to produce a dynamic gesture. Formal symbolic reasoning, on the other hand, was associated with slightly higher rates of static gesture. Verbal reasoning was associated with a mix of dynamic and static gestures.

Discussion

The results presented here support our hypothesis that dynamic representational gesture is widespread and recurrent in collaborative mathematical practice. Moreover, dynamism in gesture is associated with fictive motion in speech. Certain mathematical lexical items were particularly amenable to co-production with dynamic gesture, while others were most often treated statically. Overall, these results demonstrate that dynamic, metaphorical gesture is not restricted to pedagogical or elementary mathematical contexts, but is an essential ingredient in expert mathematical practice.

Investigating the nature of the cognitive processes undergirding mathematical cognition and gesture production requires the deployment of quantitative methods. Goldin-Meadow and her associates have successfully employed statistical methods, but these studies have been limited to simple arithmetic reasoning [1, 7, 10, 9]. Contrary to the results of the present study, they concluded that gestures have *propositional* mental representations. However, they elicited gesture during the explanation of a simple arithmetic task, and the elicited gesture consisted almost entirely of deictic gestures anchored to algebraic blackboard inscriptions. Our own previous research on dynamic gestures in advanced mathematics has shown that the explanation of advanced mathematics is often accompanied by dynamic, metaphorical gestures, but these studies were largely qualitative and opportunistic [21, 22]. This study is, to the best of our knowledge, the first to investigate quantitatively co-speech gesture during expert mathematical practice.

The present study only coded representational gestures, purposefully disregarding environmentally coupled deictic gestures. Did this unnecessarily ignore an essential component of mathematical gestural behaviour? Pointing at the graph of a function, for instance, was not coded, even if the hand dynamically swept along the curve of the graph. This aspect of the coding scheme represented a conscious decision to sacrifice breadth of analysis for experimental traction and focus. The coding scheme successfully discriminated between mathematical gestures that expressed mathematical content (e.g., “The function is increasing”), and meta-mathematical gestures that conveyed something about the local practice of mathematics (e.g. “Let’s move the diagram a bit higher on the blackboard”). While deictic gestures are undeniably an important part of mathematical practice, particularly when that practice involves collaboration at a blackboard [28, 7], such gestures are referentially ambiguous, anchored either to the inscription itself or to the mathematical entity represented by that inscription [12]. By restricting our attention to representational gestures, we avoided this complication. Future research will explore the ways in which environmentally coupled gestures elaborate meaning and direct local practices.

How are we to understand the various ways in which participants used gesture to represent mathematical concepts? Kendon [13] distinguishes three varieties of gestural representation: enactment, depiction, and modeling. In *enactment*, the motor action is meant to reproduce some features of the activity being represented. For example, gesturing in the air as if you were drawing a graph would be an enactment of the act of drawing. None of the twelve participants produced enactments that were not directly coupled to the environment (e.g. re-tracing a graph with the chalk barely off the blackboard), and thus no enactments were coded in this study. In *depiction*, the gesture “creates an object in the air” [13, p.160], often using the index finger to trace an object’s shape. Example 3 above, where the participant uses a sequence of staccato beat gestures to delimit a region in space, and the first frame of Example 4, in which the handshape stamps out a region “to the left,” are both depictions of numerical intervals that exploit the metaphorical mapping between number and space [23]. In *modeling*, the gesturing body part stands in for another object, as when a fist represents a stone. Both specimens of dynamic gesture above, Examples 1 and 2, involve the modelling of a mathematical concept — and, in particular, of a fictive trajector. In Example 1, the motion of the hands is modeling the dynamic increase of the sequence, and the hands themselves are standing in for the trajector of that fictive motion, perhaps understood as some element of the sequence. In Example 2, the hand is modeling the trajector that travels along the path of a function. For both these examples, the hands' index fingers indicate the direction of the trajectors, constituting, in a way, a vector representation of the trajector's fictive motion.

Critics of Cognitive Linguistics often object that metaphorical expressions in mathematical discourse may be "dead metaphors," expressions that once reflected underlying psychological processes but that are now entirely conventional [20, 21]. Certainly, mathematical co-speech gesture exhibits particular signs of conventionalization. In the two examples of gestures co-produced with the lexical affiliate *increasing*, both participants exhibited a marked rightward hand trajectory, mirroring the contingent fact that, historically, graphs of functions have been drawn from left to right. Is this evidence that dynamic gestures are entirely conventional, emblems that stand for common mathematical notions? Hardly. Although the direction of a numerical-spatial association may have a historical or cultural origin [23], this does not deny its cognitive reality. Indeed, the very process of associating number and space may require the cognitive mechanisms of conceptual blending and metaphor [23], and a similar conceptual association between number and space has been shown to have cognitive reality in the context of arithmetic reasoning [18, 27, 15]. This shows that even a near-universal construal requires individual cognitive elaboration and has measurable implications for reasoning. Furthermore, as we saw above, the phrase "to the left" was treated dynamically in one context, and statically in another. If this phrase were merely a conventional way of referring to certain intervals — as when we say the negative numbers are "to the left" of zero — then we would not expect a co-produced dynamic gesture under any circumstances. To the contrary, gesture in this case was dynamic, as it was in many others. The presence of dynamism in gesture, therefore, is evidence that these metaphorical expressions are not "dead," but have a cognitive reality.

Since the construction of mathematical diagrams is necessarily a dynamic process, involving the tracing of chalk across a blackboard, one might wonder if the dynamism of gesture is a mere echo of the dynamism of the inscriptive motion. Certainly, the motion that inevitably accompanies the creation of a mathematical diagram is probably a factor in the historical and developmental origin of fictive motion for such notions as continuity, function, and limit. The contemporary dynamism of mathematical thought, however, is incredibly robust. Mathematical discourse is rife with fictive motion in the absence of diagrams altogether, as demonstrated by corpus studies of purely formal analysis textbooks [25, 24]. In Example 2 above, the participant produces a dynamic gesture after writing a series of static inequalities; the blackboard is covered with static notations, with not a diagram in sight, and yet his gesture reveals the dynamism of his understanding of increasing sequences. While a graph of a function is visible in Example 1, the participants have not recently attended to the diagram; they are discussing and symbolically representing the limit of the function. Moreover, in Example 1, the participant's gesture moves orthogonally to the orientation of the diagram. The dynamism evinced in gesture is not merely an echo of inscriptive dynamism, therefore, but evidence for participants' dynamic construals.

Although certain dynamic lexical items were more often co-produced with dynamic gesture, this gestural dynamism was not mandatory. Indeed, dynamic lexical items were sometimes paired with static gesture, and in those cases where dynamic gesture was absent, the cognitive reality of the fictive motion remains an open question.

CONCLUSION

Contrary to the traditional view of mathematics — in which concepts acquire their meaning from formal definitions — the results of this study provide quantitative evidence for the claim that certain mathematical concepts are inherently metaphorical and dynamic. Participants' gesture production corroborated earlier results derived using the methods of Cognitive Linguistics [21, 22]. While previous studies of mathematical gesture production have been largely descriptive and opportunistic, this study attempted to quantify the prevalence of dynamic gesture during mathematical practice. Furthermore, this is one of the first studies to investigate gesture during advanced mathematical reasoning; previous studies focused on pedagogy or very simple mathematical problems.

Goldin-Meadow and her colleagues have shown that, in the context of elementary arithmetic, gesture can help us infer a speaker's solution strategy [5]. We found that certain modes of reasoning were associated with a preponderance of dynamic gestures, suggesting that, even in advanced mathematics, gesture may provide insight into a speaker's reasoning strategy. A speaker's topic and mode of reasoning, however, did not perfectly predict their gesture production. When speaking of a concept that exhibits fictive motion, when exactly is a speaker apt to gesture dynamically? Further research is required to identify the circumstances under which mathematical practice exploits various multimodal resources, including but not limited to co-speech gesture.

Traditional, foundationalist accounts of mathematics have focused exclusively on the *products* of mathematical activity, while ignoring the rich local practices that contribute to and, arguably, constitute the real content of mathematics. Kendon [14, p. 357] argues that a similar myopia leads to a purely formalist account of *language*, where "what is transferred to paper is abstracted away from what is actually done within an enacted utterance," and as a result essential aspects of communicative practice such as "intonation, voice quality [...] not to mention kinesis," or gesture, are ignored. In attending exclusively to formal, disembodied theorems and proofs — products of the sedimentation of local mathematical practices — the study of mathematics has ignored the rich meaning-making practices of flesh-and-blood mathematicians, and collapsed multi-agent and multimodal practice [11] into a single idealized agent working within a single written modality. In order to account for the exceptional traits of mathematics — objectivity, necessity, precision, stability – we must remember that, "Of course, in one sense, mathematics is a body of knowledge, but still it is also an *activity*" [31, §349].

The received wisdom in the Philosophy of Mathematics, therefore, is largely ahistorical, focusing on the products of mathematics while ignoring the activity of mathematics – its history and practice. The present study is a direct response to this ahistorical tradition and a contribution to an emergent study of mathematics that does justice to mathematicians and mathematical activity. This recent turn towards practice has multiplied the modalities through which we can study the nature of mathematics. By including gesture in that analysis, we take the mathematician's body seriously as a semiotic resource in the creation and communication of mathematical content.

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REFERENCES

[1] Martha Alibali, Miriam Bassok, Karen Olseth Solomon, Sharon Syc, and Susan Goldin-Meadow, ‘Illuminating mental representations through speech and gesture’, *Psychological Science*, **10**, 327–333, (1999).

[2] A. Cienki, ‘Metaphoric gestures and some of their relations to verbal metaphoric counterparts’, in *Discourse and Cognition: Bridging the Gap*, ed., J. P. Koenig, Stanford, CA, (1998). CSLI.

[3] L. D. Edwards, ‘Gestures and conceptual integration in mathematical talk’, *Educational Studies in Mathematics*, **70**(2), 127–142, (2009).

[4] Gilles Fauconnier and Mark Turner, *The Way We Think*, Basic Books, 2002.

[5] Philip Garber and Susan Goldin-Meadow, ‘Gesture offers insight into problem-solving in adults and children’, *Cognitive Science*, **26**, 817–831, (2002).

[6] D. Gentner and Dr. R. Gentner, ‘Flowing waters or teeming crowds: Mental models of electricity’, in *Cognitive functions: Classic readings in representation and reasoning*, eds., D. Gentner and A.L. Stevens, 99–129, Greenwich University Press, (1983).

[7] Susan Goldin-Meadow, Susan Wagner Cook, and Zachary Mitchell, ‘Gesturing gives children new ideas about math’, *Psychological Science*, **20**, (2009).

[8] Susan Goldin-Meadow, San Kim, and Melissa Singer, ‘What the teacher’s hands tell the student’s mind about math’, *Journal of Educational Psychology*, **91**, 720–730, (1999).

[9] Susan Goldin-Meadow, Howard Nusbaum, Spencer D. Kelly, and Susan Wagner, ‘Explaining math: Gesturing lightens the load’, *Psychological Science*, **12**, 516–522, (2001).

[10] Susan Goldin-Meadow and Susan Wagner, ‘How our hands help us learn’, *TRENDS in Cognitive Sciences*, **9**, 234–241, (2005).

[11] Charles Goodwin, ‘Action and embodiment within situated human interaction’, *Journal of Pragmatics*, **32**, 1489–1522, (2000).

[12] Edwin Hutchins and Leysia Palen, ‘Constructing meaning from space, gesture, and speech’, in *Discourse, Tools, and Reasoning: Situated Cognition and Technologically Supported Environments*, eds., L.B. Resnick, R. Saljo, C. Pontecorvo, and B. Burge, chapter 1, 22–40, Springer-Verlag, Germany, (1998).

[13] Adam Kendon, *Gesture: Visible Action as Utterance*, Cambridge University Press, 2004.

[14] Adam Kendon, ‘Some reflections on the relationship between ’gesture’ and ’sign”, *Gesture*, (2008).

[15] André Knops, Bertrand Thirion, Edward M. Hubbard, Vincent Michel, and Stanislas Dehaene, ‘Recruitment of an area involved in eye movements during mental arithmetic’, *Science*, **324**, 1583–1585, (2009).

[16] George Lakoff, ‘The contemporary theory of metaphor’, in *Metaphor and Thought*, ed., Andrew Ortony, Cambridge University Press, 2 edn., (1992).

[17] George Lakoff and Rafael Núñez, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*, Basic Books, 2000.

[18] Koleen McCrink, Stanislas Dehaene, and Ghislaine Dehaene-Lambertz, ‘Moving along the number line: Operational mementum in nonsymbolic arithmetic’, *Perception and Psychophysics*, **69**, 1324–1333, (2007).

[19] David McNeill, *Hand and mind*, University of Chicago Press, 1992.

[20] Gregory Murphy, ‘On metaphoric representation’, *Cognition*, **60**, 173–204, (1996).

[21] Rafael Núñez, ‘Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics’, in *18 Unconventional Essays on the Nature of Mathematics*, ed., Reuben Hersh, chapter 9, Springer, (2006).

[22] Rafael Núñez, ‘Mathematics, the ultimate challenge to embodiment: Truth and the grounding of axiomatic systems’, in *Handbook of Cognitive Science*, eds., O. Calvo and T. Gomilla, chapter 17, 333–353, Elsevier, (2008).

[23] Rafael Núñez, ‘No innate number line in the human brain’, *Journal of Cross-Cultural Psychology*, (in press).

[24] Rafael Núñez, L. D. Edwards, and J. F. Matos, ‘Embodied cognition as grounding for situatedness and context in mathematics education’, *Educational Studies in Mathematics*, **39**, 45–65, (1999).

[25] Rafael Núñez and George Lakoff, ‘What did Weierstrass really define? The cognitive structure of natural and epsilon-delta continuity’, *Mathematical Cognition*, **4**, 85–101, (1998).

[26] Rafael Núñez and Eve Sweetser, ‘With the future behind them: Convergent evidence from aymara language and gesture in the crosslinguistic comparison of spatial construals of time’, *Cognitive Science*, **30**, 401–450, (2006).

[27] Michal Pinhas and Martin Fischer, ‘Mental movements without magnitude? A study of spatial biases in symbolic arithmetic’, *Cognition*, **109**, 408–415, (2008).

[28] Nathaniel Smith, *Gesture and beyond*, Unpublished manuscript, University of California, Berkeley, 2003.

[29] Keith Taber, Tom de Trafford, and Teresa Quail, ‘Conceptual resources for constructing the concepts of electricity: the role of models, analogies and imagination’, *Physics Education*, **41**, 155–160, (2006).

[30] Leonard Talmy, *Towards a Cognitive Semantics*, volume 1, chapter 3: How Language Structures Space, The MIT Press, 2000.

[31] Ludwig Wittgenstein, *Philosophical Investigations*, Wiley-Blackwell, 4th edn., 2009.